

TROPICAL MULTIPLIER SEQUENCES

JENS FORSGÅRD

ABSTRACT. We prove that the diagonal operator defined by a positive sequence preserves tropical and central indices if and only if the sequence is log-concave. In particular we obtain an elementary proof of that such an operator preserves the set of sign-independently real-rooted polynomials if and only if the sequence is log-concave.

RÉSUMÉ. Nous démontrons que l'opérateur diagonal défini par une séquence positive préserve les index tropicales et centrales si et seulement si la séquence est log-concave. En particulier nous obtenons une démonstration élémentaire du fait qu'un tel opérateur préserve l'ensemble de polynômes à racines réelles, indépendamment du signe, si et seulement si la séquence est log-concave.

1. RESULTS

Consider the polynomial ring $\mathbb{R}[z]$ consisting of all real univariate polynomials. Let $\gamma = \{\gamma_n\}_{n=0}^\infty$ be a sequence of real numbers, to which we associate the diagonal operator $T_\gamma: \mathbb{R}[z] \rightarrow \mathbb{R}[z]$ defined by $z^n \mapsto \gamma_n z^n$, for $n = 0, 1, \dots$, and extended to $\mathbb{R}[z]$ by linearity. The sequence γ is said to be a multiplier sequence if T_γ preserves the set of real-rooted polynomials, see [1] for background information and applications.

We borrow the following notation from Wiman–Valiron theory, see, e.g., [2]. A non-negative integer m is said to be a *central index* of the polynomial

$$f(z) = \sum_{n=0}^d a_n z^n$$

if there exists a number $z_m \geq 0$ such that

$$(1) \quad |a_m| z_m^m \geq \sum_{n \neq m} |a_n| z_m^n.$$

Condition (1) has recently appeared in the context of amoebas, see, e.g., [5]. To relate (1) to real-rootedness, we recall that a real polynomial is said to be sign-independently real-rooted if any polynomial obtained by arbitrary sign changes of its coefficients is real-rooted, see [3].

Proposition 1. *A real polynomial f of degree d is sign-independently real-rooted if and only if each index $n = 0, \dots, d$ is a central index of f .*

In this article, we discuss sequences γ that preserve the set of central indices. In addition, we introduce the following notion. A non-negative integer m is said to be a *tropical index* of f if there exists a number $z_m \geq 0$ such that

$$(2) \quad |a_m| z_m^m \geq \max_{n \neq m} |a_n| z_m^n.$$

Notice that (2) is the analogue of (1) when the right hand side of (1) is interpreted as a tropical sum. A polynomial f of degree d is said to be *tropically real-rooted* if and only if each index $n = 0, \dots, d$ is a tropical index of f .

As the definition of central and tropical indices only depend on the moduli $|a_n|$, they are immediately extended to complex polynomials. However, for simplicity, we will henceforth assume that $a_n \geq 0$ for all n , and we will consider only positive sequences γ . Such a sequence is said to be *log-concave* if $\gamma_n^2 \geq \gamma_{n-1}\gamma_{n+1}$ for all n . In [3] it was proven, using discriminant amoebas, that the diagonal operator T_γ associated to the sequence γ preserves the set of sign-independently real-rooted polynomials if and only if γ is log-concave. For this reason, log-concave sequences are said to be *multiplier sequences of the third kind* (we will say that γ is a *tropical multiplier sequence*).

Definition 1. A sequence γ is said to be a tropical (resp. central) index preserver if for each polynomial f the set of tropical (resp. central) indices of f is a subset of the set of tropical (resp. central) indices of $T_\gamma[f]$.

Our main results are as follows.

Theorem 1. *A positive sequence γ is a tropical index preserver if and only if it is log-concave.*

Theorem 2. *A positive sequence γ is a central index preserver if and only if it is log-concave.*

As a corollary, we obtain an elementary proof of [3, Theorem 1] as requested in [3, Problem 2].

Corollary 1. *A positive sequence γ preserves the set of sign-independently real-rooted polynomials if and only if it is log-concave.*

Finally, let us state our main lemma.

Lemma 1. *A positive sequence γ is log-concave if and only if for each d the polynomial*

$$P_\gamma(z) = \sum_{n=0}^d \gamma_n z^n$$

is tropically real-rooted.

Using Lemma 1, we could rephrase Theorems 1 and 2 in a manner similar to the classical result of Pólya and Schur [4]. Namely, a positive sequence γ is a central and tropical index preserver if and only if the *tropical symbol* $P_\gamma(z)$ is tropically real-rooted.

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3. PROOFS

Proof of Lemma 1. Assume first that γ is log-concave. For each $m \geq 1$ define z_m by $z_m = \sqrt{\gamma_{m-1}/\gamma_{m+1}}$. Then,

$$\frac{z_{m+1}}{z_m} = \frac{\gamma_m}{\sqrt{\gamma_{m-1}\gamma_{m+1}}} \frac{\gamma_{m+1}}{\sqrt{\gamma_m\gamma_{m+2}}} \geq 1,$$

so that $\{z_m\}_{m=1}^\infty$ is a non-decreasing sequence of positive real numbers. Further more,

$$\frac{\gamma_m z_m^m}{\gamma_{m-1} z_m^{m-1}} = \frac{\gamma_m z_m^m}{\gamma_{m+1} z_m^{m+1}} = \frac{\gamma_m}{\sqrt{\gamma_{m-1} \gamma_{m+1}}} \geq 1.$$

Since both binomials $\gamma_n z^n - \gamma_{n+1} z^{n+1}$ and $\gamma_n z^n - \gamma_{n-1} z^{n-1}$ have exactly one positive real root, we conclude that $\gamma_n z_m^n \geq \gamma_{n+1} z_m^{n+1}$ if $n \geq m$ and that $\gamma_n z_m^n \geq \gamma_{n-1} z_m^{n-1}$ if $n \leq m$. Hence,

$$\gamma_m z_m^m \geq \max_{n \neq m} \gamma_n z_m^n.$$

For the converse, assume that γ is not log-concave. That is, there exists an index m for which $\gamma_m^2 < \gamma_{m-1} \gamma_{m+1}$. Then, for $z \geq 0$,

$$\gamma_m z^m < \sqrt{\gamma_{m-1} z^{m-1} \gamma_{m+1} z^{m+1}} \leq \max(\gamma_{m-1} z^{m-1}, \gamma_{m+1} z^{m+1}).$$

In particular, m is not a tropical index of $P_\gamma(z)$. \square

Proof of Theorem 1. Assume first that γ is log-concave. Let m be a tropical index of f , and let $z_m \geq 0$ be such that

$$a_m z_m^m \geq \max_{n \neq m} a_n z_m^n.$$

By Lemma 1 we can find a ζ_m such that

$$\gamma_m \zeta_m^m \geq \max_{n \neq m} \gamma_n \zeta_m^n.$$

Then

$$\gamma_m a_m (z_m \zeta_m)^m = \gamma_m z_m^m a_m \zeta_m^m \geq \gamma_n z_m^n a_n \zeta_m^n$$

for all n . Hence, m is a tropical index of $T_\gamma[f]$.

For the converse, it suffices to consider the polynomials $1 + z + \dots + z^d$, which is tropically real-rooted for all d , and use Lemma 1. \square

Proof of Theorem 2. Assume first that γ is log-concave, and let ζ_m be as in the proof of Theorem 1. Let m be a central index of f , and let z_m be such that

$$a_m z_m^m \geq \sum_{n \neq m} a_n z_m^n.$$

Then,

$$\gamma_m a_m (z_m \zeta_m)^m \geq \sum_{n \neq m} \gamma_m \zeta_m^m a_n z_m^n \geq \sum_{n \neq m} \gamma_n \zeta_m^n a_n z_m^n,$$

implying that m is a central index of $T_\gamma[f]$.

For the converse, assume that $\gamma_m^2 < \gamma_{m-1} \gamma_{m+1}$, and consider the action of T_γ on the trinomial $z^{m-1} + 2z^m + z^{m+1}$. \square

Proof of Proposition 1. To prove the *only if*-part, it suffices to choose z_m as the mean of the two positive roots of the polynomial

$$|a_m| z^m - \sum_{n \neq m} |a_n| z^n,$$

which exists by assumption. For the *if*-part, choose arbitrary signs of the coefficients of f . We note that the criterion (1) implies that

$$\operatorname{sgn}(f(z_m)) = \operatorname{sgn}(a_m z_m^m) = \operatorname{sgn}(a_m),$$

for $z > 0$. Using additionally Descartes' rule of signs, we conclude that the number of positive roots of f is equal to the number of sign changes in the sequence $\{a_n\}_{n=0}^d$. Similarly, the number of negative roots of f is equal to the number of sign changes in the

sequence $\{a_n(-1)^n\}_{n=0}^d$. As $a_n \neq 0$ for each n , these two numbers sums up to d , implying that $f(z)$ is real-rooted. Since the signs of the coefficients were chosen arbitrary, we are done. \square

Proof of Corollary 1. It follows from Proposition 1 that a positive sequence preserves the set of sign-independently real-rooted polynomials if and only if it preserves central indices, and it follows from Theorem 2 that a positive sequence preserves central indices if and only if it is log-concave. \square

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DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, SE-106 91 STOCKHOLM, SWEDEN.

E-mail address: jensf@math.su.se